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BRIEF COMMUNICATION

TWO SPHERES ROTATING PERPENDICULAR TO AN INTERFACE

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1. INTRODUCTION

In a recent paper, Ranger & O'Neill (1978) consider the Stokes flow problem of a sphere rotating slowly in the presence of a plane interface. Bispherical coordinates are used, and the torque on the sphere is calculated in the form of an infinite series. The authors remark that the more general problem of two rotating spheres, one in each phase, can be treated by the same method. This note addresses this two sphere problem.

2. FORMULATION OF PROBLEM IN STOKES FLOW

Letting $j = 1, 2$, assume sphere S_j lies in a fluid of viscosity μ_j, with $x = 0$ as the fluid interface; the fluid having viscosity μ_1, μ_2 occupying the half space $x > 0$, $x < 0$, respectively. The line of centres of the spheres is perpendicular to the fluid interface. Sphere S_j has radius a_j, is distance d_j from the interface, and rotates slowly about the line of centres (axis of symmetry) with angular velocity ω_j; inertia terms are neglected.

Using the notation of Ranger & O'Neill (1978), the equations are set up in terms of the variable V_j, where the fluid velocity $\mathbf{q}_j = V_j(x, \rho)/\rho\hat{\phi}$, (x, ρ, ϕ) are cylindrical polar coordinates, $\hat{\phi}$ is a unit vector perpendicular to the azimuthal plane. It is thus required to solve the boundary value problem:

$$L_{-1}(V_j) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial}{\partial \rho}\right) V_j = 0; \qquad j = 1, \qquad x > 0; \qquad j = 2, \qquad x < 0; \qquad [1]$$

with

$$V_1 = V_2, \qquad \mu_1 \frac{\partial V_1}{\partial x} = \mu_2 \frac{\partial V_2}{\partial x} \quad \text{on} \quad x = 0;$$
$$V_j = \omega_j \rho^2 \quad \text{on} \quad S_j; \qquad [2]$$
$$\mathbf{q}_j \to 0 \quad \text{at infinity}.$$

Defining bispherical coordinates by

$$\rho = \frac{c \sin \eta}{\cosh \xi - \cos \eta}, \qquad x = \frac{c \sinh \xi}{\cosh \xi - \cos \eta}, \qquad c > 0,$$

the spheres are given by $\xi_j = \alpha_j$, with $\alpha_1 > 0$, $\alpha_2 < 0$,

$$a_j = c|\text{cosech}\, \alpha_j|, \qquad d_j = a_j \cosh \alpha_j.$$

Suitable solutions of [1] are

$$V_j = c(\cosh\xi - \cos\eta)^{-1/2}\sum_{n=1}^{\infty}[A_n^{(j)}\cosh p\xi + B_n^{(j)}\sinh p\xi]P_n^1(\cos\eta)\sin\eta, \qquad [3]$$

where $n+1/2$ has been denoted by p. The interface conditions in [2] requiring continuity of velocity and stress give $A_n^{(1)} = A_n^{(2)}$, $B_n^{(1)} = \mu B_n^{(2)}$ where $\mu = \mu_2/\mu_1$; and the boundary conditions on S_j require $V_j = \omega_j c^2 \sin^2\eta\,(\cosh\alpha_j - \cos\eta)^{-2}$ on $\xi = \alpha_j$. Using a well-known series involving the Legendre functions, the boundary value problem reduces to two equations in $A_n^{(2)}$, $B_n^{(2)}$, with solution

$$A_n^{(2)} = A_n^{(1)} = 2\surd(2)c(\omega_1 e^{-p\alpha_1}\sinh p\alpha_2 - \mu\omega_2 e^{p\alpha_2}\sinh p\alpha_1)/\Delta, \qquad [4]$$

$$B_n^{(2)} = B_n^{(1)}/\mu = 2\surd(2)c(\omega_2 e^{p\alpha_2}\cosh p\alpha_1 - \omega_1 e^{-p\alpha_1}\cosh p\alpha_2)/\Delta, \qquad [5]$$

where $\Delta = \cosh p\alpha_1 \sinh p\alpha_2 - \mu\cosh p\alpha_2 \sinh p\alpha_1$. At this stage it is interesting to note that if $\mu = 1$ the formulas agree with those of Jeffery (1915) who considered the motion of a single viscous fluid generated by the rotation of two spheres.

3. TORQUE ON A SPHERE

Using Jeffery's formula for the torque $-T_j\hat{k}$, (where $\hat{k}$ is a unit vector perpendicular to the plane of motion) with

$$T_j = 2\pi\mu_j\int_0^{\pi}\rho^3\frac{\partial}{\partial\xi}\left(\frac{V_j}{\rho^2}\right)d\eta \quad \text{on} \quad \xi = \alpha_j;$$

gives the torque $-T_1\hat{k}$ on sphere S_1 with

$$\tau_1 = \frac{T_1}{8\pi\mu_1\omega_1 c^3} = \sum_{n=1}^{\infty}\frac{2n(n+1)}{\Delta}e^{-p\alpha_1}[(1+\mu\omega)\sinh p\alpha_2 + \mu(\omega-1)\cosh p\alpha_2], \qquad [6]$$

where $\omega = \omega_2/\omega_1$. The torque on sphere S_2 can be found similarly. τ_1 can be evaluated numerically for given values of the sphere parameters α_j, and given viscosity ratio μ, and angular velocity ratio ω (which can take either sign). For small values of α_j it would be desirable for rapid convergence to use a Watson transform as Davis *et al.* (1976). When $\alpha_2 \to -\infty$ the infinite series [6] for the torque reduces to that found by Ranger & O'Neill (1978). Their numerical calculations show that, in their one sphere problem, τ_1 increases as μ increases.

Several interesting results emerge in the special case of two spheres with equal radii $c|\text{cosech}\,\alpha|$, at $c|\coth\alpha|$ on each side of the interface, where $\alpha = \alpha_1 = -\alpha_2$, $\alpha > 0$. From [6] the torque on S_1 is given by

$$\tau_1 = \sum_{n=1}^{\infty}\frac{2n(n+1)}{\sinh 2p\alpha}\left[1 - \lambda e^{-2p\alpha} - \frac{2\mu}{1+\mu}\omega e^{-2p\alpha}\right],$$

where $\lambda = (1-\mu)/(1+\mu)$. Thus, using an infinite series for cosech^3,

$$\tau_1 = \sum_{m=0}^{\infty}\left[\text{cosech}^3(2m+1)\alpha - \left(\lambda + \frac{2\mu\omega}{1+\mu}\right)\text{cosech}^3 2(m+1)\alpha\right]. \qquad [7]$$

For $\mu \neq 0$ the influence of the rotation of sphere S_2 for $\omega > 0$ is thus seen to reduce the value of

τ_1. A zero torque gives the value of ω corresponding to the angular velocity with which S_1 rotates if left free. Jeffery (1915) gives a table of such values in the case of a uniform fluid ($\mu = 1$, $\lambda = 0$). As an illustration of changing the viscosity ratio μ, some numerical values for τ_1 are given below in table 1. The value of ω is fixed at 30; and the value of α fixed at 1.0, so that the ratio of the distance of the centre of each sphere from the interface to the radius of the sphere is 1.5431. For small μ, the torque is positive, decreases to zero at $\mu = 0.9576$, and continues to decrease as μ increases beyond this value.

Table 1. Values of τ_1 against μ, for fixed $\alpha = 1$ (so $d/a = 1.5431$), $\omega = 30$

μ	0	0.1	0.2	0.5	1.0	2.0	5.0	10.0	20.0
τ_1	0.5961	0.4853	0.3930	0.1899	−0.0132	−0.2163	−0.4194	−0.5117	−0.5645

Consider now some given angular velocity ratios for this case of two equal spheres.

(i) $\omega = 1$, the spheres rotate with equal angular velocity.

$$\tau_1 = \sum_{m=0}^{\infty} (-1)^m \operatorname{cosech}^3 (m+1)\alpha,$$

and the stress is zero on the interface, see [5]. This result is identical with the limit $\mu \to 0$ in [7], thus the sphere S_1 feels no influence of the rotating sphere S_2 which is in an inviscid fluid.

(ii) $\omega = -1$, the spheres rotate with angular velocities equal in magnitude but opposite in direction.

$$\tau_1 = \sum_{m=0}^{\infty} \left[\operatorname{cosech}^3 (2m+1)\alpha - \frac{(1-3\mu)}{(1+\mu)} \operatorname{cosech}^3 2(m+1)\alpha\right],$$

giving an interface with zero velocity only in the special case $\mu = 1$, see [4].

(iii) $\omega = -1/\mu$, the spheres rotate in opposite directions with magnitudes inversely proportional to the viscosity of the fluids in which they are immersed, $-\mu_2\omega_2 = \mu_1\omega_1$.

$$\tau_1 = \sum_{m=0}^{\infty} \operatorname{cosech}^3 (m+1)\alpha,$$

and the velocity is zero at the interface. However, the interface is not a stress-free surface, see Davis *et al.* (1975). This value of torque is identical with the limiting case of [6] when $\alpha_2 \to 0$, $\omega_2 \to 0$, that is the fluid with viscosity μ_2 is replaced by a rigid fixed plane.

(iv) As a special case of (i) consider a dumbbell of two equal spheres rotating with the same angular velocity and their plane of tangency at the fluid interface. In the limit as $\alpha_1 \to 0$

$$\tau_1 = \frac{3}{4}\zeta(3) \operatorname{cosech}^3 \alpha,$$

and there is a similar expression for τ_2, thus the torque on the dumbbell agrees with the expression of Schneider *et al.* (1973).

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